B.Sc. Semester-V Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 52116 **Course Code : SH/MTH/503/DSE-1**

Course Title : Linear Programming

OR

Theory of Equations

OR

Point Set Topology

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks. Candidates are required to give their answers in their own words as far as practicable.

Notations and synbols have their usual meaning.

(Linear Programming)

1. Answer any **five** questions: 2×5=10

   a) Write down an example of bounded convex set and unbounded convex set for each.

   b) Write the dual of

$$\text{Minimize } z = x_1 - 3x_2 + 2x_3$$
$$\text{subject to } 3x_1 - x_2 + 2x_3 \le 7,$$
$$-2x_1 + 4x_2 \le 12,$$
$$-4x_1 + 3x_2 + 8x_3 \le 10,$$
$$x_i \ge 0,\ i = 1, 2, 3.$$

[Turn Over]

   c) When does an LPP admit an alternative basic optimal solution? Answer in the context of Simplex Method.

   d) Obtain an initial basic feasible solution to the following transportation problem using the North-West Corner method.

DESTINATION

ORIGINS				a_i
	5	1	8	12
	2	4	0	14
	3	6	7	4
b_j	9	10	11	

   e) In a Primal-Dual Problem

      i) When the dual objective function is unbounded?

      ii) When both the objective functions are unbounded?

   f) What do you mean by a balanced transportation problem?

   g) Write down mathematical formulation of an assignment problem.

   h) Use dominance to reduce the pay off matrices and solve the game:

2	3	$\frac{1}{2}$
$\frac{3}{2}$	2	0
$\frac{1}{2}$	1	1

2. Answer any **four** questions: 5×4=20

a) Following is the starting tableau of an LPP by the simplex method (Big M Method) (for a maximization problem), in an incomplete form.

			c_j						
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6
-M	a_6	x_6	1	1	1	-1			
0	a_4	x_4	4	2	1	0			
0	a_5	x_5	15	5	8	0			
$(z_j - c_j)$				-M-3	-M-2	M	0	0	0

i) Complete the objective row and the tableau.

ii) Write down the LPP in its standard form from the tableau.

iii) Write down the actual problem.

iv) Find the departing and the entering vectors and write down the next tableau. 1+1+1+2

b) By solving the dual, show that there exists an unbounded optimal solution to the primal

Maximize $z = 4x_1 + 2x_2$

subject to $x_1 + x_2 \geq 3$

$x_1 - x_2 \geq 2$

$x_1, x_2 \geq 0$

c) i) Find the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ using the simplex method.

ii) Write the maximum number of basic solutions to a set of m simultaneous equations in n unknowns $(n \geq m)$. 4+1

d) Find the optimal solution and the corresponding cost of transportation in the transportation problem.

	D_1	D_2	D_3	D_4	a_i
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
O_4	22	35	25	41	

e) Five operators to be assigned to five machines. The assignment cost are given below. Operator A cannot operate machine III and operator C cannot operate machine IV. Find the optimal assignment schedule and minimum cost of assignment.

MACHINE

OPERATORS	I	II	III	IV	V
A	5	5	–	2	6
B	7	4	2	3	4
C	9	3	5	–	3
D	7	2	6	7	2
E	6	5	7	9	1

f) Find the optimal assignment to find the minimum cost for the assignment problems with the following cost matrix:

	I	II	III	IV	V
A	3	8	2	10	3
B	8	7	2	9	7
C	6	4	2	7	5
D	8	4	2	3	5
E	9	10	6	9	10

3. Answer any **one** question: $10\times1=10$

a) i) Verify that the dual of the dual is primal for the following LPP:

Maximize $z = 2x_1 - 3x_2$

subject to $-x_1 + x_2 \le 1$

$x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

ii) Show that the set of all convex combinations of a finite number of points $x_1, x_2, ..., x_n$ is a convex set.

iii) Graphically find the solution of the following LPP:

Maximize $z = 50x_1 + 15x_2$

subject to $5x_1 + x_2 \le 100$

$x_1 + x_2 \le 50$

$x_1, x_2 \ge 0$ 5+2+3=10

b) i) Define separating and supporting Hyperplanes. Consider the rectangle whose vertices are (0,0), (1,0), (1,1), (0,1). Express the point (0.4, 0.6) as a convex combination of the four extreme points of the convex set defined by the rectangle and its interior.

ii) What is the 'Rectangular Game'? Show that, for the 2×2 game with pay-off matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

I) It is non-strictly determined if a<b, a<c, d<b and d<c **OR** a>b, a>c, d>b and d>c.

II) It has no saddle point if a<d<b<c .

iii) In a game of matching coins with two players A and B, suppose A wins one unit of value when there are two heads, wins nothing when there are two tails and loses $\frac{1}{2}$ unit of value when there is one head and one tail. Determine the pay off matrix and the optimal strategies for both the players.

3+4+3=10

(Theory of Equations)

1. Answer any **five** questions: 2×5=10

a) Show that x^2+x+1 is a factor of $x^{10}+x^5+1$.

b) Find the remainder when the polynomial x^3-3x^2+4x-3 is divided by $\left(x-\sqrt{2}\right)$.

c) Apply Descartes' rule of signs, to find the nature of the roots of the equation $x^4+qx^2+rx-s=0$, where $q>r>s>0$.

d) Solve the equation $x^3-6x^2+3x+10=0$ considering its roots in arithmetic progression.

e) If α, β, γ are the roots of the equation $x^3-px-q=0$, then find the value of $\sum\frac{1}{\alpha\beta}$.

f) Find the equation whose roots are reciprocals of the roots of the equation $x^3+px^2+qx+r=0$.

g) Find the special roots of the equation $x^6-1=0$.

h) Write down the Sturm's functions for the equation $x^3+3x+1=0$.

2. Answer any **four** questions: 5×4=20

a) i) Let $f(x)$ be a polynomial of degree $n>2$. Show that the remainder in the division of $f(x)$ by $(x-1)(x-2)$ is given by

$$(x-1)f(2)-(x-2)f(1).$$

ii) If $f(x)$ is a polynomial, then prove that $(x-a)$ is a factor of $f(x)$ if and only if $f(a)=0$. 3+2

b) i) If the equation $ax^3+3bx^2+3cx+d=0$ has two equal roots, then show that $(bc-ad)^2=4(b^2-ac)(c^2-ad)$.

ii) If ax^3+bx^2+cx+d is divisible by x^2+l^2, then show that $ad=bc$. 3+2

c) If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4+px^3+qx^2+rx+s=0$, then find the values of

i) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$ and

ii) $\alpha^2\beta+\beta^2\gamma+\gamma^2\delta+\delta^2\alpha$ 3+2

d) Solve the reciprocal equation

$$x^5-5x^4+9x^3-9x^2+5x-1=0.$$ 5

e) Prove that the roots of the following equation are all real:

$$\frac{1}{x+1}+\frac{1}{x+2}+...+\frac{1}{x+n}=\frac{1}{x}.$$

f) Solve the following equation by Ferrari's method:

$$x^4+12x-5=0.$$ 5

3. Answer any **one** question: 10×1=10

a) i) If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4-x^3+x^2-x+1=0$, then find the values of

A) $(\alpha+1)(\beta+1)+(\gamma+1)(\delta+1)$ and

B) $(\alpha^2+4)(\beta^2+4)+(\gamma^2+4)(\delta^2+4)$.

ii) If x^4+px^2+qx+r has a factor of the form $(x-\alpha)^3$, then show that $8p^2+27q^2=0$ and $p^2+12r=0$.

iii) Let α, β, γ be the roots of the equation $x^3+qx+r=0$. Then find the equation whose roots are $1+\frac{1}{\alpha}$, $1+\frac{1}{\beta}$, $1+\frac{1}{\gamma}$.

(3+2)+3+2

b) i) If α, β, γ are the roots of the cubic $x^3-9x+9=0$ then prove that

$$(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)=\pm 27.$$

ii) Show that the equation $x^3-3x+k=0$ has three distinct real roots, if $-2<x<2$.

iii) Remove the third term of the equation:

$$x^4+8x^3+19x^2+12x-5=0.$$

5+3+2

(Point Set Topology)

1. Answer any **five** questions: 2×5=10

   a) Give an example to show that a T_2-space need not be T_3-space.

   b) Is the interval (0, 1) compact? Justify your answer.

   c) Define a basis and a subbasis for topology.

   d) If X = {a, b, c} with the given topology by $\tau_1 = \{x, \phi, \{a\}, \{b, c\}\}$ and Y = {p, q, r} where the topology is $\tau_2 = \{y, \phi, \{r\}, \{p, q\}\}$, can you give an example of a mapping $f: X \to Y$ which is not continuous?

   e) Is the space $\mathbb{R}^n$ locally compact? Justify your answer.

   f) Define a first-countable space with an example.

   g) If A and B are connected subsets of a space X such that $A \cap B \neq \phi$. Prove that $A \cup B$ is connected.

   h) Define the subspace topology with an example.

2. Answer any **four** questions: 5×4=20

   a) If A is a subset of a topological space then prove that

      i) $\overline{A}$ is the smallest closed set containing A.

      ii) A is closed iff $\overline{A} = A$. 2+3=5

   b) Prove that every closed and bounded subset of R is compact. 5

   c) Prove that a second countable space is first countable but the converse need not true. 5

   d) i) Prove that a closed subset of a compact topological space is compact.

      ii) Prove that the continuous image of a compact topological space is compact. 2+3=5

   e) Define a separable space. Show that every second countable space is separable. 2+3=5

   f) i) Define T_0, T_1, T_2 spaces.

      ii) Give examples to show that a T_0 space need not be T_1 and a T_1 space need not be T_2. 2+3=5

3. Answer any **one** question: 10×1=10

   a) i) Show that a mapping f of a space X into a space Y is continuous iff

$$\left(f^{-1}(B)\right)^0 \supset f^{-1}\left(B^0\right).$$

ii) Let (X, τ) be a topological space and let A, B be any two subsets of X. Then show that $A \subset B \Rightarrow \overline{A} \subset \overline{B}$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

iii) Show that every metric space is T_3-space.

4+4+2=10

b) i) Prove that a subset of $\mathbb{R}$ is connected if it is an interval.

ii) Show that the continuous image of connected space is connected.

iii) Prove that intersection of any arbitrary collection of topologies on X is also a topology on X.

4+3+3=10

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